

# THE $C^*$ -ALGEBRA OF THE EXPONENTIAL FUNCTION

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**ABSTRACT.** The complex exponential function  $e^z$  is a local homeomorphism and gives therefore rise to an étale groupoid and a  $C^*$ -algebra. We show that this  $C^*$ -algebra is simple, purely infinite, stable and classifiable by K-theory, and has both K-theory groups isomorphic to  $\mathbb{Z}$ . The same methods show that the  $C^*$ -algebra of the anti-holomorphic function  $\overline{e^z}$  is the stabilisation of the Cuntz-algebra  $\mathcal{O}_3$ .

## 1. INTRODUCTION

The crossed product of a locally compact Hausdorff space by a homeomorphism has been generalised to local homeomorphisms in the work of Renault, Deaconu and Anantharaman-Delaroche, [Re],[De], [An]. In many cases the algebra is both simple and purely infinite, and can be determined by the use of the Kirchberg-Phillips classification result. The purpose with the present note is to demonstrate how methods and results about the iteration of complex holomorphic functions can be used for this purpose. This will be done by determining the  $C^*$ -algebra of an entire holomorphic function  $f$  when  $f'(z) \neq 0$  and  $\#f^{-1}(f(z)) \geq 2$  for all  $z \in \mathbb{C}$ , and the Julia set  $J(f)$  of  $f$  is the whole complex plane  $\mathbb{C}$ . A prominent class of functions with these properties is the family  $\lambda e^z$  where  $\lambda > \frac{1}{e}$ . These functions commute with complex conjugation and we can therefore use the same methods to determine the  $C^*$ -algebra of the anti-holomorphic function  $\overline{f}$ . The results are as stated in the abstract for  $f(z) = e^z$ .

## 2. THE $C^*$ -ALGEBRA OF A LOCAL HOMEOMORPHISM

**2.1. The definition.** We describe in this section the construction of a  $C^*$ -algebra from a local homeomorphism. It was introduced in increasing generality by J. Renault [Re], V. Deaconu [De] and Anantharaman-Delaroche [An].

Let  $X$  be a second countable locally compact Hausdorff space and  $\varphi : X \rightarrow X$  a local homeomorphism. Set

$$\Gamma_\varphi = \{(x, k, y) \in X \times \mathbb{Z} \times X : \exists n, m \in \mathbb{N}, k = n - m, \varphi^n(x) = \varphi^m(y)\}.$$

This is a groupoid with the set of composable pairs being

$$\Gamma_\varphi^{(2)} = \{((x, k, y), (x', k', y')) \in \Gamma_\varphi \times \Gamma_\varphi : y = x'\}.$$

The multiplication and inversion are given by

$$(x, k, y)(y, k', y') = (x, k + k', y') \text{ and } (x, k, y)^{-1} = (y, -k, x).$$

Note that the unit space of  $\Gamma_\varphi$  can be identified with  $X$  via the map  $x \mapsto (x, 0, x)$ . Under this identification the range map  $r : \Gamma_\varphi \rightarrow X$  is the projection  $r(x, k, y) = x$  and the source map the projection  $s(x, k, y) = y$ .

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To turn  $\Gamma_\varphi$  into a locally compact topological groupoid, fix  $k \in \mathbb{Z}$ . For each  $n \in \mathbb{N}$  such that  $n + k \geq 0$ , set

$$\Gamma_\varphi(k, n) = \{(x, l, y) \in X \times \mathbb{Z} \times X : l = k, \varphi^{k+n}(x) = \varphi^n(y)\}.$$

This is a closed subset of the topological product  $X \times \mathbb{Z} \times X$  and hence a locally compact Hausdorff space in the relative topology. Since  $\varphi$  is locally injective  $\Gamma_\varphi(k, n)$  is an open subset of  $\Gamma_\varphi(k, n + 1)$  and hence the union

$$\Gamma_\varphi(k) = \bigcup_{n \geq -k} \Gamma_\varphi(k, n)$$

is a locally compact Hausdorff space in the inductive limit topology. The disjoint union

$$\Gamma_\varphi = \bigcup_{k \in \mathbb{Z}} \Gamma_\varphi(k)$$

is then a locally compact Hausdorff space in the topology where each  $\Gamma_\varphi(k)$  is an open and closed set. In fact, as is easily verified,  $\Gamma_\varphi$  is a locally compact groupoid in the sense of [Re], i.e. the groupoid operations are all continuous, and an étale groupoid in the sense that the range and source maps are local homeomorphisms.

To obtain a  $C^*$ -algebra, consider the space  $C_c(\Gamma_\varphi)$  of continuous compactly supported functions on  $\Gamma_\varphi$ . They form a  $*$ -algebra with respect to the convolution-like product

$$f \star g(x, k, y) = \sum_{z, n+m=k} f(x, n, z)g(z, m, y)$$

and the involution

$$f^*(x, k, y) = \overline{f(y, -k, x)}.$$

To obtain a  $C^*$ -algebra, let  $x \in X$  and consider the Hilbert space  $l^2(s^{-1}(x))$  of square summable functions on  $s^{-1}(x) = \{(x', k, y') \in \Gamma_\varphi : y' = x\}$  which carries a representation  $\pi_x$  of the  $*$ -algebra  $C_c(\Gamma_\varphi)$  defined such that

$$(\pi_x(f)\psi)(x', k, x) = \sum_{z, n+m=k} f(x', n, z)\psi(z, m, x)$$

when  $\psi \in l^2(s^{-1}(x))$ . One can then define a  $C^*$ -algebra  $C_r^*(\Gamma_\varphi)$  as the completion of  $C_c(\Gamma_\varphi)$  with respect to the norm

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\|.$$

Since we assume that  $X$  is second countable it follows that  $C_r^*(\Gamma_\varphi)$  is separable. Note that this  $C^*$ -algebra can be constructed from any locally compact étale groupoid  $\Gamma$  in the place of  $\Gamma_\varphi$ , see e.g. [Re], [An]. Note also that  $C_r^*(\Gamma_\varphi)$  is the classical crossed product  $C_0(X) \rtimes_\varphi \mathbb{Z}$  when  $\varphi$  is a homeomorphism.

### 3. THE GENERALISED PIMSNER-VOICULESCU EXACT SEQUENCE

There is a six-terms exact sequence which can be used to calculate the  $K$ -theory of  $C_r^*(\Gamma_\varphi)$ . It was obtained from the work of Pimsner, [Pi], by Deaconu and Muhly in a slightly different setting in [DM]. In particular, Deaconu and Muhly require  $\varphi$  to be surjective and essentially free, but thanks to the work of Katsura in [Ka] we can now establish it for arbitrary local homeomorphisms. This generalisation will be important here because  $e^z$  is not surjective.

Consider the set

$$\Gamma_\varphi(1, 0) = \{(x, 1, y) \in \Gamma_\varphi(1) : y = \varphi(x)\},$$

which is an open subset of  $\Gamma_\varphi(1)$  and hence of  $\Gamma_\varphi$ . Set  $E_0 = C_c(\Gamma_\varphi(1, 0))$ . Note that  $f^*g \in C_c(\varphi(X)) \subseteq C_c(\Gamma_\varphi)$  when  $f, g \in E_0$ . In fact

$$f^*g(x, k, y) = \begin{cases} 0, & k \neq 0 \vee x \neq y \vee x \notin \varphi(X) \\ \sum_{z \in \varphi^{-1}(x)} \overline{f(z, 1, x)} g(z, 1, x), & k = 0 \wedge x = y \in \varphi(X). \end{cases}$$

It follows that the closure  $E$  of  $E_0$  in  $C_r^*(\Gamma_\varphi)$  is a Hilbert  $C_0(X)$ -module with an  $C_0(X)$ -valued inner product  $\langle \cdot, \cdot \rangle$  defined such that  $\langle f, g \rangle = f^*g$ ,  $f, g \in E$ . Since  $C_0(X)E \subseteq E$  we can consider  $E$  as a  $C^*$ -correspondence over  $C_0(X)$  in the obvious way, cf. Definition 1.3 of [Ka]. Let  $\iota : C_0(X) \rightarrow C_r^*(\Gamma_\varphi)$  and  $t : E \rightarrow C_r^*(\Gamma_\varphi)$  be the inclusion maps. Then  $(\iota, t)$  is an injective representation of the  $C^*$ -correspondence  $E$  in the sense of Katsura, cf. Definitions 2.1 and 2.2 of [Ka].

Let  $\mathbb{K}(E)$  be the  $C^*$ -algebra of adjointable operators on  $E$  generated by the elementary operators  $\Theta_{f,g}$ ,  $f, g \in E$ , where  $\Theta_{f,g}(k) = f \langle g, k \rangle$ .

**Lemma 3.1.**  $C_0(X) \subseteq \mathbb{K}(E)$ .

*Proof.* Since  $\Theta_{f,g}(k) = fg^*k$  when  $f, g, k \in E$  it suffices to show that the elements of  $C_0(X)$  of the form  $fg^*$  for some  $f, g \in E$  span a dense subspace of  $C_0(X)$ . Let  $U$  be an open subset of  $X$  where  $\varphi$  is injective and consider a non-negative function  $h \in C_c(X)$  supported in  $U$ . Then

$$W = \{(x, 1, \varphi(x)) : x \in U\}$$

is an open subset of  $\Gamma_\varphi(1, 0)$  and we define  $f \in E_0$  such that  $\text{supp } f \subseteq W$  and  $f(x, 1, \varphi(x)) = \sqrt{h(x)}$ ,  $x \in U$ . Then  $h = \Theta_{f,f}$  and we are done.  $\square$

It follows that  $(\iota, t)$  is covariant in the sense of [Ka], cf. Proposition 3.3 and Definition 3.4 of [Ka], and there is therefore an associated  $*$ -homomorphism  $\rho : \mathcal{O}_E \rightarrow C_r^*(\Gamma_\varphi)$ .

**Proposition 3.2.**  $\rho : \mathcal{O}_E \rightarrow C_r^*(\Gamma_\varphi)$  is an isomorphism.

*Proof.* By construction  $C_r^*(\Gamma_\varphi)$  carries an action  $\beta$  by the circle  $\mathbb{T}$  defined such that

$$\beta_\lambda(f)(x, k, y) = \lambda^k f(x, k, y)$$

when  $f \in C_c(\Gamma_\varphi)$ . This is the *gauge action*. This action ensures that  $(\iota, t)$  admits a gauge action in the sense of [Ka]. By Theorem 6.4 of [Ka] it suffices therefore to show that  $C_r^*(\Gamma_\varphi)$  is generated, as a  $C^*$ -algebra, by  $C_0(X)$  and  $E$ .

Let  $\mathcal{A}$  be the  $*$ -subalgebra of  $C_c(\Gamma_\varphi)$  generated by  $C_c(X)$  and  $E_0$ . Let  $k \in \mathbb{N}$ . We claim that  $C_c(\Gamma_\varphi(k, 0)) \in \mathcal{A}$ . Since  $C_c(\Gamma_\varphi(0, 0)) = C_c(X)$  and  $E_0 = C_c(\Gamma_\varphi(1, 0))$  it suffices to prove this when  $k \geq 2$ . To this end it suffices, since  $C_c(X) \subseteq \mathcal{A}$ , to consider an open subset  $U$  of  $X$  on which  $\varphi^k$  is injective and show that any non-negative continuous function  $h$  compactly supported in  $\{(x, k, \varphi^k(x)) : x \in U\}$  is in  $\mathcal{A}$ . To this end, let  $f_j \in E_0$  be supported in

$$\{(y, 1, \varphi(y)) : y \in \varphi^{j-1}(U)\}$$

and satisfy that  $f_j(\varphi^{j-1}(x), 1, \varphi^j(x)) = h(x, k, \varphi^k(x))^{\frac{1}{k}}$  for all  $x \in U$ . Then  $h = f_1 f_2 \cdots f_k$  and hence  $h \in \mathcal{A}$ .

We will next prove by induction that  $C_c(\Gamma_\varphi(k, n)) \subseteq \mathcal{A}$  for all  $n \in \mathbb{N}$ . The assertion holds when  $n = 0$  as we have just shown, so assume that it holds for  $n$ . To show that  $C_c(\Gamma_\varphi(k, n+1)) \subseteq \mathcal{A}$ , let  $U$  and  $V$  be open subsets in  $X$  such that  $\varphi^n$  is injective on both  $U$  and  $V$ . It suffices to consider a continuous function  $h$  compactly supported in  $\Gamma_\varphi(k, n+1) \cap (U \times \{k\} \times V)$  and show that  $h \in \mathcal{A}$ . Note that  $W = \Gamma_\varphi(k, n) \cap (\varphi(U) \times \{k\} \times \varphi(V))$  is open in  $\Gamma_\varphi(k, n)$  and that we can define  $\tilde{h} : W \rightarrow \mathbb{C}$  such that  $\tilde{h}(x, k, y) = h(x', k, y')$ , where  $x' \in U, y' \in V$  and  $\varphi(x') = x, \varphi(y') = y$ . Then  $\tilde{h}$  is continuous and has compact support in  $W$ ; in fact, the support is the image of the support,  $K$ , of  $h$  under the continuous map  $\Gamma_\varphi(k, n+1) \ni (x, k, y) \mapsto (\varphi(x), k, \varphi(y)) \in \Gamma_\varphi(k, n)$ . Hence  $\tilde{h} \in \mathcal{A}$  by assumption. Note that  $r(K)$  is a compact subset of  $U$  and  $s(K)$  a compact subset of  $V$ . Let  $a \in C_c(X)$  be supported in  $U$  such that  $a(x) = 1, x \in r(K)$ , and  $b \in C_c(X)$  be supported in  $V$  such that  $b(x) = 1, x \in s(K)$ . Define  $\tilde{a}, \tilde{b} \in C_c(\Gamma_\varphi(1, 0)) = E_0$  with supports in  $\{(x, 1, \varphi(x)) : x \in U\}$  and  $\{(x, 1, \varphi(x)) : x \in V\}$ , respectively, such that  $\tilde{a}(x, 1, \varphi(x)) = a(x)$  when  $x \in U$  and  $\tilde{b}(x, 1, \varphi(x)) = b(x)$  when  $x \in V$ . Since  $h = \tilde{a}\tilde{h}\tilde{b}^*$  we conclude that  $h \in \mathcal{A}$ . Thus  $C_c(\Gamma_\varphi(k, n)) \subseteq \mathcal{A}$  for all  $k \geq 0, n \geq 0$ . Since  $C_c(\Gamma_\varphi(-k, n))^* = C_c(\Gamma_\varphi(n, n-k))$  when  $n \geq k \geq 0$ , we conclude that  $\mathcal{A} = C_c(\Gamma_\varphi)$ .  $\square$

By combining Proposition 3.2 and Lemma 3.1 with Theorem 8.6 of [Ka] we obtain the following.

**Theorem 3.3.** (Deaconu and Muhly, [DM]) *Let  $[E] \in KK(C_0(X), C_0(X))$  be the element represented by the embedding  $C_0(X) \subseteq \mathbb{K}(E)$ . There is an exact sequence*

$$\begin{array}{ccccc} K_0(C_0(X)) & \xrightarrow{\text{id}_* - [E]^*} & K_0(C_0(X)) & \xrightarrow{\iota_*} & K_0(C_r^*(\Gamma_\varphi)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\Gamma_\varphi)) & \xleftarrow{\iota_*} & K_1(C_0(X)) & \xleftarrow{\text{id}_* - [E]_*} & K_1(C_0(X)) \end{array}$$

#### 4. SIMPLE PURELY INFINITE $C^*$ -ALGEBRAS FROM ENTIRE FUNCTIONS WITHOUT CRITICAL POINTS IN THE JULIA SET

Throughout this section  $f : \mathbb{C} \rightarrow \mathbb{C}$  is an entire function of degree at least 2; i.e. either a polynomial of degree at least 2 or a transcendental function. An  $n$ -periodic point  $z \in \mathbb{C}$  is *repelling* when  $|(f^n)'(z)| > 1$ . The *Julia set*  $J(f)$  of  $f$  can then be defined as the closure of the repelling periodic points. Although this is not the standard definition it emphasises one of the properties that will be important here. Others are

- i)  $J(f)$  is non-empty and perfect, and
- ii)  $J(f)$  is totally  $f$ -invariant, i.e.  $f^{-1}(J(f)) = J(f)$ .

We refer to the survey by Bergweiler, [Be], for the proof of these properties.

Let  $\mathcal{E}(f)$  denote the set of points  $x \in \mathbb{C}$  such that  $f^{-1}(x) = \{x\}$ . For example, when  $f(z) = 2ze^z$  the point 0 will be in  $\mathcal{E}(f) \cap J(f)$ .

**Lemma 4.1.**  $\#\mathcal{E}(f) \leq 1$ .

*Proof.* Let  $x, y \in \mathcal{E}(f)$  and assume for a contradiction that  $x \neq y$ . Since  $J(f)$  is infinite,  $J(f) \setminus \{x, y\}$  is not empty. Let  $U$  be an open subset of  $\mathbb{C}$  such that

$$U \cap (J(f) \setminus \{x, y\}) \neq \emptyset. \quad (4.1)$$

Then  $\bigcup_{i=0}^{\infty} f^i(U \setminus \{x, y\})$  is open, non-empty,  $f$ -invariant and does not contain  $\{x, y\}$ . It follows therefore from Montel's theorem, cf. Theorem 3.7 in [Mi], that  $f^n, n \in \mathbb{N}$ , is a normal family when restricted to  $\bigcup_{i=0}^{\infty} f^i(U \setminus \{x, y\})$ . This contradicts (4.1) since  $U \setminus \{x, y\}$  contains a repelling periodic point.  $\square$

The set  $J(f) \setminus \mathcal{E}(f)$  is locally compact in the relative topology inherited from  $\mathbb{C}$  and  $f^{-1}(J(f) \setminus \mathcal{E}(f)) = J(f) \setminus \mathcal{E}(f)$ . If we now assume that  $f'(z) \neq 0$  when  $z \in J(f)$ , it follows that the restriction

$$F : J(f) \setminus \mathcal{E}(f) \rightarrow J(f) \setminus \mathcal{E}(f)$$

of  $f$  to  $J(f) \setminus \mathcal{E}(f)$  is a local homeomorphism on the second countable locally compact Hausdorff space  $J(f) \setminus \mathcal{E}(f)$ . Note that  $F$  is not always surjective - it is not when  $f(z) = e^z$ .

Following [An] we say that an étale groupoid  $\Gamma$  with range map  $r$  and source map  $s$  is *essentially free* when the points  $x$  of the unit space  $\Gamma^0$  for which the isotropy group  $s^{-1}(x) \cap r^{-1}(x)$  is trivial (i.e. only consists of  $\{x\}$ ) is dense in  $\Gamma^0$ . For the groupoid  $\Gamma_f$  of a local homeomorphism  $f : X \rightarrow X$  this occurs if and only if  $\{x \in X : f^i(x) = x\}$  has empty interior for all  $i \in \mathbb{N}$ .

We say that  $\Gamma$  is *minimal* when there is no open non-empty subset  $U$  of  $\Gamma^0$ , other than  $\Gamma^0$ , which is  $\Gamma$ -invariant in the sense that  $r(\gamma) \in U \Leftrightarrow s(\gamma) \in U$  for all  $\gamma \in \Gamma$ . This holds for the groupoid  $\Gamma_f$  if and only if the full orbit  $\bigcup_{i,j \in \mathbb{N}} f^{-i}(f^j(x))$  is dense in  $X$  for all  $x \in X$ .

Finally, we say that  $\Gamma$  is *locally contracting* when every open non-empty subset of  $\Gamma^0$  contains an open non-empty subset  $V$  with the property that there is an open bisection  $S$  in  $\Gamma$  such that  $\overline{V} \subseteq s(S)$  and  $\alpha_S^{-1}(\overline{V}) \subsetneq V$  when  $\alpha_S : r(S) \rightarrow s(S)$  is the homeomorphism defined by  $S$ , cf. Definition 2.1 of [An] (but note that the source map is denoted by  $d$  in [An]).

**Lemma 4.2.** *Assume that  $f'(z) \neq 0$  for all  $z \in J(f)$ . Then  $\Gamma_F$  is minimal, essentially free and locally contracting in the sense of [An].*

*Proof.* To show that  $\Gamma_F$  is essentially free we must show that

$$\{z \in J(f) \setminus \mathcal{E}(f) : F^i(z) = z\} \quad (4.2)$$

has empty interior in  $J(f) \setminus \mathcal{E}(f)$  for all  $i \in \mathbb{N}$ . Assume that  $U$  is open in  $\mathbb{C}$  and that  $U \cap J(f) \setminus \mathcal{E}(f)$  is a non-empty subset of (4.2). Since  $J(f)$  is perfect it follows that every point  $z_0$  of  $U \cap J(f) \setminus \mathcal{E}(f)$  is the limit of a sequence from

$$\{z \in \mathbb{C} : f^i(z) = z\} \setminus \{z_0\}.$$

Since  $f$  is entire it follows that  $f^i(z) = z$  for all  $z \in \mathbb{C}$ , contradicting that  $J(f) \neq \emptyset$ . Hence  $\Gamma_F$  is essentially free.

To show that  $\Gamma_F$  is minimal, consider an open subset  $U \subseteq \mathbb{C}$  such that

$$U \cap J(f) \setminus \mathcal{E}(f) \neq \emptyset. \quad (4.3)$$

Let  $W = \bigcup_{i,j \in \mathbb{N}} f^{-j}(f^i(U \setminus \mathcal{E}(f)))$ . Since  $W$  is open (in  $\mathbb{C}$ ), non-empty,  $f$ -invariant and has non-trivial intersection with  $J(f)$  it follows from Montel's theorem, cf. Theorem 3.7 in [Mi], that  $\mathbb{C} \setminus W$  contains at most one element. Note that this element must be in  $\mathcal{E}(f)$  because  $W$  and hence also  $\mathbb{C} \setminus W$  is totally  $f$ -invariant. It

follows therefore that  $W \cap J(f) \setminus \mathcal{E}(f) = J(f) \setminus \mathcal{E}(f)$ . Hence

$$\bigcup_{i,j \in \mathbb{N}} f^{-j} (f^i(U \cap J(f) \setminus \mathcal{E}(f))) = W \cap J(f) = J(f) \setminus \mathcal{E}(f),$$

proving that  $\Gamma_F$  is minimal.

To show that  $\Gamma_F$  is locally contracting consider an open subset  $U$  of  $\mathbb{C}$  such that  $U \cap J(f) \setminus \mathcal{E}(f) \neq \emptyset$ . There is then a repelling periodic point  $z_0 \in U \cap J(f) \setminus \mathcal{E}(f)$ . There is therefore an  $n \in \mathbb{N}$ , a positive number  $\kappa > 1$  and an open neighbourhood  $W \subseteq U$  of  $z_0$  such that  $f^n(z_0) = z_0$ ,  $f^n$  is injective on  $W$  and

$$|f^n(y) - z_0| \geq \kappa |y - z_0| \quad (4.4)$$

for all  $y \in W$ . Let  $\delta_0 > 0$  be so small that

$$\{y \in \mathbb{C} : |y - z_0| \leq \delta_0\} \subseteq f^n(W) \cap W. \quad (4.5)$$

$z_0$  is not isolated in  $J(f) \setminus \mathcal{E}(f)$  since  $J(f)$  is perfect. There is therefore an element  $z_1 \in J(f) \setminus \mathcal{E}(f)$  such that  $0 < |z_1 - z_0| < \delta_0$ . Choose  $\delta$  strictly between  $|z_1 - z_0|$  and  $\delta_0$  such that

$$\kappa |z_1 - z_0| > \delta. \quad (4.6)$$

Set  $V_0 = \{y \in \mathbb{C} : |y - z_0| < \delta\}$ . Then

$$\overline{V_0} \cap J(f) \setminus \mathcal{E}(f) \subsetneq f^n(V_0 \cap J(f) \setminus \mathcal{E}(f)). \quad (4.7)$$

Indeed, if  $|y - z_0| \leq \delta$  (4.5) implies that there is a  $y' \in W$  such that  $f^n(y') = y$  and then (4.4) implies that  $|y' - z_0| < \delta$ . Since  $y' \in J(f) \setminus \mathcal{E}(f)$  when  $y \in J(f) \setminus \mathcal{E}(f)$  it follows that  $\overline{V_0} \cap J(f) \setminus \mathcal{E}(f) \subseteq f^n(V_0 \cap J(f) \setminus \mathcal{E}(f))$ . On the other hand, it follows from (4.6) and (4.4) that  $f^n(z_1) \notin \overline{V_0}$ . This shows that (4.7) holds. Then

$$S = \{(z, n, f^n(z)) \in \Gamma_F(n, 0) : z \in V_0\}$$

is an open bisection in  $\Gamma_F$  such that  $\overline{V_0} \cap J(f) \setminus \mathcal{E}(f) \subseteq s(S)$  and

$$\alpha_{S^{-1}}(\overline{V_0} \cap J(f) \setminus \mathcal{E}(f)) \subsetneq V_0 \cap J(f) \setminus \mathcal{E}(f).$$

Then  $V = V_0 \cap J(f) \setminus \mathcal{E}(f)$  is an open subset of  $U \cap J(f) \setminus \mathcal{E}(f)$  such that  $\alpha_{S^{-1}}(\overline{V}) \subsetneq V$ . This shows that  $\Gamma_F$  is locally contracting.  $\square$

**Corollary 4.3.** *The  $C^*$ -algebra  $C_r^*(\Gamma_F)$  is simple and purely infinite.*

*Proof.* By Theorem 4.16 of [Th] simplicity is a consequence of the minimality and essential freeness of  $\Gamma_F$ . Pure infiniteness follows from Proposition 2.4 in [An] because  $\Gamma_F$  is essentially free and locally contracting.  $\square$

## 5. THE $C^*$ -ALGEBRA OF THE EXPONENTIAL FUNCTION

For the statement of the next theorem, which is the main result of the note, recall that the separable, stable, simple purely infinite  $C^*$ -algebras which satisfy the universal coefficient theorem (UCT) of Rosenberg and Schochet, [RS], is exactly the class of  $C^*$ -algebras known from the Kirchberg-Phillips results, [Ph], to be classified by their  $K$ -theory groups alone.

**Theorem 5.1.** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire transcendental function such that*

- i)  $f'(z) \neq 0 \ \forall z \in \mathbb{C}$ ,
- ii) *the Julia set  $J(f)$  of  $f$  is  $\mathbb{C}$ , and*
- iii)  $\#f^{-1}(f(x)) \geq 2$  *for all  $x \in \mathbb{C}$ .*

Then  $C_r^*(\Gamma_f)$  is the separable stable simple purely infinite  $C^*$ -algebra which satisfies the UCT, and  $K_0(C_r^*(\Gamma_f)) \simeq K_1(C_r^*(\Gamma_f)) \simeq \mathbb{Z}$ .

*Proof.* First observe that  $f$  is a local homeomorphism because it is holomorphic with no critical points by assumption i). Hence  $C_r^*(\Gamma_f)$  is defined. As we pointed out above the separability of  $C_r^*(\Gamma_f)$  follows because  $\mathbb{C}$  has a countable base for its topology. It follows from Proposition 3.2, Lemma 3.1 and Proposition 8.8 of [Ka] that  $C_r^*(\Gamma_f)$  satisfies the UCT. Since iii) implies that  $\mathcal{E}(f) = \emptyset$  it follows from Corollary 4.3 that  $C_r^*(\Gamma_f)$  is simple and purely infinite. Since  $C_r^*(\Gamma_f)$  is not unital (because  $\mathbb{C}$  is not compact), it follows from Theorem 1.2 of [Z] that  $C_r^*(\Gamma_f)$  is stable.

It remains now only to calculate the  $K$ -theory of  $C_r^*(\Gamma_f)$ . We use Theorem 3.3 for this and we need therefore to determine the action on  $K$ -theory of the  $KK$ -element  $[E]$ . Let  $\Delta$  be a small open disc centered at  $0 \in \mathbb{C}$  such that  $f$  is injective on  $\overline{\Delta}$ . Set  $V = f(\Delta)$  and let  $i : C_0(\Delta) \rightarrow C_0(\mathbb{C})$  and  $j : C_0(V) \rightarrow C_0(\mathbb{C})$  denote the natural embeddings. Define  $\psi_f : C_0(\Delta) \rightarrow C_0(V)$  such that  $\psi_f(g) = g \circ f^{-1}$ . It is easy to see that

$$i^*[E] = j_*[\psi_f] = [j \circ \psi_f]$$

in  $KK(C_0(\Delta), C_0(\mathbb{C}))$ . To proceed we apply Schoenflies' theorem to get a homeomorphism  $F : \mathbb{C} \rightarrow \mathbb{C}$  extending  $f : U \rightarrow V$ . Note that  $F$  must be orientation preserving since  $f$  is. It follows therefore that  $F$  is isotopic to the identity, cf. Theorem 2.4.2 on page 92 in [L]. This shows that  $j \circ \psi_f$  is homotopic to  $i$  and we conclude therefore that  $[j \circ \psi_f] = [i] = i^*[\text{id}_{C_0(\mathbb{C})}]$ . Since  $i^* : KK(C_0(\mathbb{C}), C_0(\mathbb{C})) \rightarrow KK(C_0(\Delta), C_0(\mathbb{C}))$  is an isomorphism it follows that  $[E] = [\text{id}_{C_0(\mathbb{C})}]$ . The conclusion that  $K_0(C_r^*(\Gamma_f)) \simeq K_1(C_r^*(\Gamma_f)) \simeq \mathbb{Z}$  follows now straightforwardly from the generalised Pimsner-Voiculescu exact sequence of Theorem 3.3.  $\square$

The function  $f(z) = \lambda e^z$  clearly satisfies assumptions i) and iii) of Theorem 5.1 when  $\lambda \neq 0$ . Furthermore, when  $\lambda > \frac{1}{e}$  it is shown in [De] that also assumption ii) holds, extending the result of Misiurewicz, [M], dealing with the case  $\lambda = 1$ .

## 6. THE $C^*$ -ALGEBRA OF $e^{\overline{z}}$

Let  $\mathbb{K}$  be the  $C^*$ -algebra of compact operators on an infinite-dimensional separable Hilbert space.

**Theorem 6.1.** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire transcendental function such that*

- i)  $f'(z) \neq 0 \ \forall z \in \mathbb{C}$ ,
- ii) *the Julia set  $J(f)$  of  $f$  is  $\mathbb{C}$ ,*
- iii)  $\#f^{-1}(f(x)) \geq 2$  *for all  $x \in \mathbb{C}$ , and*
- iv)  $\overline{f(z)} = f(\overline{z})$ ,  $z \in \mathbb{C}$ .

*Define  $\overline{f} : \mathbb{C} \rightarrow \mathbb{C}$  such that  $\overline{f}(z) = \overline{f(z)}$ . Then  $C_r^*(\Gamma_{\overline{f}}) \simeq \mathcal{O}_3 \otimes \mathbb{K}$  where  $\mathcal{O}_3$  is the Cuntz-algebra with  $K_0(\mathcal{O}_3) \simeq \mathbb{Z}_2$  and  $K_1(\mathcal{O}_3) = 0$ , cf. [C].*

*Proof.*  $C_r^*(\Gamma_{\overline{f}})$  is separable and satisfies the UCT for the same reason that  $C_r^*(\Gamma_f)$  has these properties. Since  $J(f^2) = J(f) = \mathbb{C}$  and  $\mathcal{E}(f^2) = \emptyset$  by iii) we conclude from Lemma 4.2 that  $\Gamma_{f^2}$  is minimal, essentially free and locally contracting. Since  $\Gamma_{f^2} \subseteq \Gamma_{\overline{f}}$  it follows that  $\Gamma_{\overline{f}}$  is minimal and locally contracting. Furthermore, by using that

$$\{z \in \mathbb{C} : \overline{f}^i(z) = z\} \subseteq \{z \in \mathbb{C} : f^{2i}(z) = z\},$$

it follows also that  $\Gamma_{\bar{f}}$  is essentially free because  $\Gamma_{f^2}$  is. As in the proof of Corollary 4.3 we conclude now that  $C_r^*(\Gamma_{\bar{f}})$  is simple and purely infinite. Finally, since  $\bar{f}$  is orientation reversing the calculation of the  $K$ -theory in the proof of Theorem 5.1 now yields the conclusion that  $[E] = -[\text{id}_{C_0(\mathbb{C})}]$ , leading to the result that  $K_0(C_r^*(\Gamma_{\bar{f}})) \simeq \mathbb{Z}_2$  while  $K_1(C_r^*(\Gamma_{\bar{f}})) = 0$ . Hence the theorem of Zhang, [Z], and the Kirchberg-Phillips classification theorem, Theorem 4.2.4 of [Ph], imply that  $C_r^*(\Gamma_{\bar{f}}) \simeq \mathcal{O}_3 \otimes \mathbb{K}$ .  $\square$

## REFERENCES

- [An] C. Anantharaman-Delaroche, *Purely infinite  $C^*$ -algebras arising from dynamical systems*, Bull. Soc. Math. France **125** (1997), 199–225.
- [Be] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1993), 151–188.
- [C] J. Cuntz,  *$K$ -theory for certain  $C^*$ -algebras*, Ann. of Math. **113** (1981), 181–197.
- [De] V. Deaconu, *Groupoids associated with endomorphisms*, Trans. Amer. Math. Soc. **347** (1995), 1779–1786.
- [DM] V. Deaconu and P.S. Muhly,  *$C^*$ -algebras associated with branched coverings*, Proc. Amer. Math. Soc. **129** (2000), 1077–1086.
- [D] R.L. Devaney, *Cantor bouquets, explosions, and Knaster continua: Dynamics of complex exponentials*, Publicacions Matemàtiques **43** (1999), 27–54.
- [Ka] T. Katsura, *On  $C^*$ -algebras associated with  $C^*$ -correspondences*, J. Func. Analysis **217** (2004), 366–401.
- [L] T. Lawson, *Topology: A geometric approach*, Oxford University Press, 2003.
- [Mi] J. Milnor, *Dynamics in one complex variable*, Annals of Mathematics Studies, 160, Princeton University Press, Princeton, NJ, 2006.
- [M] M. Misiurewicz, *On iterates of  $e^z$* , Ergodic Th. & Dynam. Sys. **1** (1981), 103–106.
- [Ph] N. C. Phillips, *A classification theorem for nuclear purely infinite simple  $C^*$ -algebras*, Doc. Math. **5** (2000), 49–114.
- [Pi] M.V. Pimsner, *A class of  $C^*$ -algebras generalizing both Cuntz-Krieger algebras and crossed products by  $\mathbb{Z}$*  in D. Voiculescu (Ed.), Free Probability Theory, Fields Inst. Communication, Vol 12, American Mathematical Society, Providence, RI, 1997, pp. 189–212.
- [Re] J. Renault, *A Groupoid Approach to  $C^*$ -algebras*, LNM 793, Springer Verlag, Berlin, Heidelberg, New York, 1980.
- [RS] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov’s generalized  $K$ -functor*, Duke J. Math. **55** (1987), 431–474.
- [Th] K. Thomsen, *Semi-étale groupoids and applications*, Annales de l’Institut Fourier **60** (2010), 759–800.
- [Z] S. Zhuang, *Certain  $C^*$ -algebras with real rank zero and their corona and multiplier algebras. Part I*, Pac. J. Math. **155** (1992), 169–197.

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